

Tables of Implications and Tautologies from Symbolic Logic

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Here are some tables of logical equivalents and implications that I have found useful over the years. Where there are classical names for things I have included them. “Isolation by Parts” is my own invention. By **tautology** I mean equivalent left and right hand side and by **implication** I mean the left hand expression implies the right hand. I use the tilde and overbar interchangeably to represent negation e.g. $\sim x$ is the same as \bar{x} . Enjoy!

Table 1: **Properties of All Two-bit Operators.** The Comm. is short for commutative and Assoc. is short for associative. Iff is short for “if and only if”.

Truth Table	Name	Comm./ Assoc.	Binary Op	And/Or/Not	Nands Only
0000	False	CA	0	0	$(a \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow a))$
0001	And	CA	$a \wedge b$	$a \wedge b$	$(a \uparrow b) \uparrow (a \uparrow b)$
0010	Minus		$b - a$	$\bar{a} \wedge b$	$(b \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow a))$
0011	B	A	b	b	b
0100	Minus		$a - b$	$a \wedge \bar{b}$	$(a \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow b))$
0101	A	A	a	a	a
0110	Xor/NotEqual	CA	$a \oplus b$	$(a \wedge \bar{b}) \vee (\bar{a} \wedge b)$ $(a \vee b) \wedge (\bar{a} \vee \bar{b})$	$(b \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow b))$
0111	Or	CA	$a \vee b$	$a \vee b$	$(a \uparrow a) \uparrow (b \uparrow b)$
1000	Nor	C	$a \downarrow b$	$\bar{a} \wedge \bar{b}$	$((a \uparrow a) \uparrow (b \uparrow b)) \uparrow ((a \uparrow a) \uparrow a)$
1001	Iff/Equal	CA	$a \leftrightarrow b$	$(a \vee \bar{b}) \wedge (\bar{a} \vee b)$ $(a \wedge b) \vee (\bar{a} \wedge \bar{b})$	$((a \uparrow a) \uparrow (b \uparrow b)) \uparrow (a \uparrow b)$
1010	Not A		\bar{a}	\bar{a}	$a \uparrow a$
1011	Imply		$a \rightarrow b$	$\bar{a} \vee b$	$(a \uparrow (a \uparrow b))$
1100	Not B		\bar{b}	\bar{b}	$b \uparrow b$
1101	Imply		$b \rightarrow a$	$a \vee \bar{b}$	$(b \uparrow (a \uparrow a))$
1110	Nand	C	$a \uparrow b$	$\bar{a} \vee \bar{b}$	$a \uparrow b$
1111	True	CA	1	1	$(a \uparrow a) \uparrow a$

Table 2: **Tautologies (Logical Identities)**

Commutative Property:	$p \wedge q \leftrightarrow q \wedge p$
	$p \vee q \leftrightarrow q \vee p$
	$p \oplus q \leftrightarrow q \oplus p$
Associative Property:	$(p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$
	$(p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$
	$(p \oplus q) \oplus r \leftrightarrow p \oplus (q \oplus r)$
Distributive Property:	$p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$
	$p \wedge (q \oplus r) \leftrightarrow (p \wedge q) \oplus (p \wedge r)$
	$p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$
	$p \vee (q \rightarrow r) \leftrightarrow (p \vee q) \rightarrow (p \vee r)$
	$p \rightarrow (q \wedge r) \leftrightarrow (p \rightarrow q) \wedge (p \rightarrow r)$
	$p \rightarrow (q \vee r) \leftrightarrow (p \rightarrow q) \vee (p \rightarrow r)$
De Morgan's Laws:	$\sim(p \wedge q) \leftrightarrow \sim p \vee \sim q$
	$\sim(p \vee q) \leftrightarrow \sim p \wedge \sim q$
	$\sim(p \oplus q) \leftrightarrow \sim p \oplus q$
	$\sim(p \oplus q) \leftrightarrow p \oplus \sim q$
	$\sim(p \rightarrow q) \leftrightarrow p \wedge \sim q$
Transposition (Contrapositive):	$p \rightarrow q \leftrightarrow \sim q \rightarrow \sim p$
	$p \oplus q \leftrightarrow \sim p \oplus \sim q$
Involution (Double Negation):	$\sim \sim p \leftrightarrow p$
Material Implication:	$p \rightarrow q \leftrightarrow \sim p \vee q$
Material Equivalence (Biconditional):	$(p \rightarrow q) \wedge (q \rightarrow p) \leftrightarrow p \leftrightarrow q$
Partial Associativity:	$p \rightarrow (q \rightarrow r) \leftrightarrow q \rightarrow (p \rightarrow r)$
Exportation:	$(p \wedge q) \rightarrow r \leftrightarrow p \rightarrow (q \rightarrow r)$
Absurdity:	$(p \rightarrow q) \wedge (p \rightarrow \sim q) \leftrightarrow \sim p$
	$(\sim p \vee q) \wedge (p \vee r) \leftrightarrow (p \wedge q) \vee (\sim p \wedge r)$
Absorption:	$(p \wedge q) \vee p \leftrightarrow p$
	$(p \vee q) \wedge p \leftrightarrow p$
Isolation by Parts (of p by q):	$(p \wedge \sim q) \vee (p \wedge q) \leftrightarrow p$
	$(p \wedge \sim q) \oplus (p \wedge q) \leftrightarrow p$
Destructive Distribution:	$p \wedge (\sim p \vee q) \leftrightarrow p \wedge q$
	$p \wedge (\sim p \oplus q) \leftrightarrow p \wedge q$
	$p \wedge (p \rightarrow q) \leftrightarrow p \wedge q$
	$p \vee (\sim p \wedge q) \leftrightarrow p \vee q$
	$p \vee (p \oplus q) \leftrightarrow p \vee q$
	$p \vee (\sim p \rightarrow q) \leftrightarrow p \vee q$
	$p \rightarrow (\sim p \vee q) \leftrightarrow p \rightarrow q$
	$p \rightarrow (\sim p \oplus q) \leftrightarrow p \rightarrow q$
	$(p \vee q) \rightarrow q \leftrightarrow p \rightarrow q$
	$(p \oplus q) \rightarrow q \leftrightarrow p \rightarrow q$
	$(p \rightarrow q) \rightarrow p \leftrightarrow p$
	$p - (q - p) \leftrightarrow p$
Right Distribution of Implies	$(p \rightarrow r) \vee (q \rightarrow r) \leftrightarrow (p \wedge q) \rightarrow r$
	$(p \rightarrow r) \wedge (q \rightarrow r) \leftrightarrow (p \vee q) \rightarrow r$
	$(p \rightarrow r) \oplus (q \rightarrow r) \leftrightarrow \sim((p \oplus q) \rightarrow r)$

Table 3: **Multiple Statement Classical Implications** These are the classical logical implications from arguments. The semicolon separates different statements in a proof. The semicolon can be replaced by a logical AND and it becomes a single true statement.

Modus Ponens:	$p \rightarrow q; p \rightarrow q$
Modus Tollens:	$p \rightarrow q; \sim q \rightarrow \sim p$
Hypothetical Syllogism:	$p \rightarrow q; q \rightarrow r \rightarrow p \rightarrow r$
Disjunctive Syllogism:	$p \vee q; \sim p \rightarrow q$
Constructive Dilemma:	$(p \rightarrow q) \wedge (r \rightarrow s); p \vee r \rightarrow q \vee s$
Destructive Dilemma:	$(p \rightarrow q) \wedge (r \rightarrow s); \sim q \vee \sim s \rightarrow \sim p \vee \sim r$
Conjunction:	$p; q \rightarrow p \wedge q$
(no name):	$p \rightarrow q \rightarrow p$

Table 4: **Single Statement Implications** Some of these are named classical logical implications and others are simply unnamed tautologous implications. Resolution is useful in eliminating a variable from an expression in conjunctive normal form. This happens in the Davis-Putnam algorithm for example.

Simplification:	$p \wedge q \rightarrow p$
Addition:	$p \rightarrow p \vee q$
Subtraction:	$p - q \rightarrow p$
Law of Resolution:	$(p \vee q) \wedge (\sim p \vee r) \rightarrow q \vee r$
Weakening:	$p \leftrightarrow q \rightarrow p \rightarrow q$
	$\sim(p \rightarrow q) \rightarrow q \rightarrow p$
	$p \rightarrow q \rightarrow p \wedge q$
	$p \rightarrow q \rightarrow (p \wedge r) \rightarrow q$
	$p \rightarrow q \rightarrow (p \wedge r) \rightarrow (q \wedge r)$
	$p \rightarrow q \rightarrow (p \vee r) \rightarrow (q \vee r)$
	$p \oplus q \rightarrow p \vee q$
	$p \wedge q \rightarrow p \vee q$

Table 5: **Equivalents in Packing and Unpacking Bit Fields** In contrast to previous sections, this section deals with operators that work only on bitstrings. Specifically, the symbols \vdash and \dashv are pack and unpack bit fields. $(p \dashv m)$ means to **unpack** the string p using mask m . For example: $(1011 \dashv 110011)$ gives 100011. The length of p must be the same as the number of 1 bits in m . $(p \vdash m)$ means to **pack** the string p using mask m . For example: $(1011011 \vdash 1100111)$ gives 10011. The length of p must be the same as the length of m . The resulting string has the same number of bits as there are 1 bits in m . An alternate implementation of this operator may pad the resulting bitstring on the left with 0 so that the result is the same length as the two operands. The results of both forms contain equal information. All logic operators are bitwise operators. Overbar is the one's complement operator. $\vec{1}$ is a string of all 1 bits. The equals sign means the bit strings are identical. The names of these theorems are my own.

Compressive Subset	$(p \vdash p) = (\vec{1} \vdash p)$
Inverse Property:	$(p \dashv m) \vdash m = p \wedge (\vec{1} \vdash m)$
Semi-Inverse Property:	$(p \vdash m) \dashv m = p \wedge m$
Associative Property:	$((p \dashv m) \dashv n) = (p \dashv (m \dashv n))$
Negation of Pack:	$\overline{p \vdash m} = (\overline{p} \vdash m) \vee (\vec{1} \vdash m)$
Negation of Unpack:	$\overline{p \dashv m} = \overline{m} \vee (\overline{p} \dashv m)$
	$\overline{p \dashv m} = \overline{m} \oplus (\overline{p} \dashv m)$
Distributive Property:	$(p \vee q) \dashv m = (p \dashv m) \vee (q \dashv m)$
	$(p \oplus q) \dashv m = (p \dashv m) \oplus (q \dashv m)$
	$(p \wedge q) \dashv m = (p \dashv m) \wedge (q \dashv m)$
	$(p \vee q) \vdash m = (p \vdash m) \vee (q \vdash m)$
	$(p \oplus q) \vdash m = (p \vdash m) \oplus (q \vdash m)$
	$(p \wedge q) \vdash m = (p \vdash m) \wedge (q \vdash m)$
Destructive Distribution:	$(m \wedge p) \vdash p = m \vdash p$
	$(m \oplus p) \vdash p = \overline{m} \vdash p$
	$(m \dashv p) \wedge p = m \dashv p$
	$(m \dashv p) \oplus p = \overline{m} \dashv p$
	$(p \dashv (m \vee n)) \vdash m = p \vdash (m \vdash (m \vee n))$