Tables of Tautologies from Symbolic Logic

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Here are some tables of logical equivalents and implications that I have found useful over the years. Where there are classical names for things I have included them. By **tautology** I mean equivalent left and right hand side and by **implication** I mean the left hand expression implies the right hand.

Table 1: **Properties of All Two-bit Operators**

Truth Table	Comm./ Assoc.	Binary Op	And/Or/Not	Nands
0000	CA	0	0	$(a \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow a))$
0001	CA	$a \wedge b$	$a \wedge b$	$(a \uparrow b) \uparrow (a \uparrow b)$
0010		$b - a$	$\overline{a} \wedge b$	$(b \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow a))$
0011	A	b	b	b
0100		$a - b$	$a \wedge \overline{b}$	$(a \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow b))$
0101	A	a	a	a
0110	CA	$a \oplus b$	$(a \wedge \overline{b}) \vee (\overline{a} \wedge b)$ $(a \vee b) \wedge (\overline{a} \vee \overline{b})$	$(b \uparrow (a \uparrow a)) \uparrow (a \uparrow (a \uparrow b))$
0111	CA	$a \vee b$	$a \vee b$	$(a \uparrow a) \uparrow (b \uparrow b)$
1000	C	$a \downarrow b$	$\overline{a} \wedge \overline{b}$	$((a \uparrow a) \uparrow (b \uparrow b)) \uparrow ((a \uparrow a) \uparrow a)$
1001	CA	$a = b$	$(a \vee \overline{b}) \wedge (\overline{a} \vee b)$ $(a \wedge b) \vee (\overline{a} \wedge \overline{b})$	$((a \uparrow a) \uparrow (b \uparrow b)) \uparrow (a \uparrow b)$
1010		$\overline{a}$	$\overline{a}$	$a \uparrow a$
1011		$a \rightarrow b$	$\overline{a} \vee b$	$(a \uparrow (a \uparrow b))$
1100		$\overline{b}$	$\overline{b}$	$b \uparrow b$
1101		$b \rightarrow a$	$a \vee \overline{b}$	$(b \uparrow (a \uparrow a))$
1110	C	$a \uparrow b$	$\overline{a} \vee \overline{b}$	$a \uparrow b$
1111	CA	1	1	$(a \uparrow a) \uparrow a$

Table 2: **Tautologies (Logical Identities)**

Commutative Property:	$p \wedge q$	$\leftrightarrow$	$q \wedge p$
	$p \vee q$	$\leftrightarrow$	$q \vee p$
	$p \oplus q$	$\leftrightarrow$	$q \oplus p$
Associative Property:	$(p \wedge q) \wedge r$	$\leftrightarrow$	$p \wedge (q \wedge r)$
	$(p \vee q) \vee r$	$\leftrightarrow$	$p \vee (q \vee r)$
	$(p \oplus q) \oplus r$	$\leftrightarrow$	$p \oplus (q \oplus r)$
Distributive Property:	$p \wedge (q \vee r)$	$\leftrightarrow$	$(p \wedge q) \vee (p \wedge r)$
	$p \wedge (q \oplus r)$	$\leftrightarrow$	$(p \wedge q) \oplus (p \wedge r)$
	$p \vee (q \wedge r)$	$\leftrightarrow$	$(p \vee q) \wedge (p \vee r)$
	$p \vee (q \rightarrow r)$	$\leftrightarrow$	$(p \vee q) \rightarrow (p \vee r)$
	$p \rightarrow (q \wedge r)$	$\leftrightarrow$	$(p \rightarrow q) \wedge (p \rightarrow r)$
	$p \rightarrow (q \vee r)$	$\leftrightarrow$	$(p \rightarrow q) \vee (p \rightarrow r)$
De Morgan's Laws:	$\sim(p \wedge q)$	$\leftrightarrow$	$\sim p \vee \sim q$
	$\sim(p \vee q)$	$\leftrightarrow$	$\sim p \wedge \sim q$
	$\sim(p \oplus q)$	$\leftrightarrow$	$\sim p \oplus q$
	$\sim(p \oplus q)$	$\leftrightarrow$	$p \oplus \sim q$
	$\sim(p \rightarrow q)$	$\leftrightarrow$	$p \wedge \sim q$
Transposition (Contrapositive):	$p \rightarrow q$	$\leftrightarrow$	$\sim q \rightarrow \sim p$
	$p \oplus q$	$\leftrightarrow$	$\sim p \oplus \sim q$
Involution (Double Negation):	$\sim\sim p$	$\leftrightarrow$	p
Material Implication:	$p \rightarrow q$	$\leftrightarrow$	$\sim p \vee q$
Material Equivalence:	$p \leftrightarrow q$	$\leftrightarrow$	$(p \rightarrow q) \wedge (q \rightarrow p)$
Partial Associativity:	$p \rightarrow (q \rightarrow r)$	$\leftrightarrow$	$q \rightarrow (p \rightarrow r)$
Exportation:	$(p \wedge q) \rightarrow r$	$\leftrightarrow$	$p \rightarrow (q \rightarrow r)$
Absurdity:	$(p \rightarrow q) \wedge (p \rightarrow \sim q)$	$\leftrightarrow$	$\sim p$
	$(\sim p \vee q) \wedge (p \vee r)$	$\leftrightarrow$	$(p \wedge q) \vee (\sim p \wedge r)$
Absorption:	$(p \wedge q) \vee p$	$\leftrightarrow$	p
	$(p \vee q) \wedge p$	$\leftrightarrow$	p
Destructive Distribution:	$p \wedge (\sim p \vee q)$	$\leftrightarrow$	$p \wedge q$
	$p \wedge (\sim p \oplus q)$	$\leftrightarrow$	$p \wedge q$
	$p \wedge (p \rightarrow q)$	$\leftrightarrow$	$p \wedge q$
	$p \vee (\sim p \wedge q)$	$\leftrightarrow$	$p \vee q$
	$p \vee (\sim p \rightarrow q)$	$\leftrightarrow$	$p \vee q$
	$p \vee (p \oplus q)$	$\leftrightarrow$	$p \vee q$
	$p \rightarrow (\sim p \vee q)$	$\leftrightarrow$	$p \rightarrow q$
	$p \rightarrow (\sim p \oplus q)$	$\leftrightarrow$	$p \rightarrow q$
	$(p \vee q) \rightarrow q$	$\leftrightarrow$	$p \rightarrow q$
	$(p \oplus q) \rightarrow q$	$\leftrightarrow$	$p \rightarrow q$
	$(p \rightarrow q) \rightarrow p$	$\leftrightarrow$	q
	$(p \rightarrow r) \vee (q \rightarrow r)$	$\leftrightarrow$	$(p \wedge q) \rightarrow r$
	$(p \rightarrow r) \wedge (q \rightarrow r)$	$\leftrightarrow$	$(p \vee q) \rightarrow r$
	$(p \rightarrow r) \oplus (q \rightarrow r)$	$\leftrightarrow$	$\sim((p \oplus q) \rightarrow r)$

Table 3: **Multiple Statement Classical Implications** These are the classical logical implications from arguments. The semicolon separates different statements in a proof. The semicolon can be replaced by a logical AND and it becomes a single true statement.

Modus Ponens:	$p \to q\,;\quad p$	$\to$	q
Modus Tollens:	$p \to q\,;\quad \sim q$	$\to$	$\sim p$
Hypothetical Syllogism:	$p \to q\,;\quad q \to r$	$\to$	$p \to r$
Disjunctive Syllogism:	$p \vee q\,;\quad \sim p$	$\to$	q
Constructive Dilemma:	$(p \to q) \wedge (r \to s)\,;\quad p \vee r$	$\to$	$q \vee s$
Destructive Dilemma:	$(p \to q) \wedge (r \to s)\,;\quad \sim q \vee \sim s$	$\to$	$\sim p \vee \sim r$
Conjunction:	$p\,;\quad q$	$\to$	$p \wedge q$

Table 4: **Single Statement Implications** Some of these are named classical logical implications and others are simply unnamed tautologous implications. Resolution is useful in eliminating a variable from an expression in conjunctive normal form. This happens in the Davis-Putnam algorithm for example.

Simplification:	$p \wedge q$	$\to$	p
Addition:	p	$\to$	$p \vee q$
Subtraction:	$p - q$	$\to$	p
Law of Resolution:	$(p \vee q) \wedge (\sim p \vee r)$	$\to$	$q \vee r$
	$\sim(p \to q)$	$\to$	$q \to p$
	$p \to q$	$\to$	$p \wedge q$
	$p \to q$	$\to$	$(p \wedge r) \to q$
	$p \to q$	$\to$	$(p \wedge r) \to (q \wedge r)$
	$p \to q$	$\to$	$(p \vee r) \to (q \vee r)$
	$p \oplus q$	$\to$	$p \vee q$
	$p \wedge q$	$\to$	$p \vee q$

Table 5: **Equivalents in Packing and Unpacking Bit Fields** In contrast to previous sections, this section deals with operators that work only on bitstrings. Specifically, the symbols $\vdash$ and $\dashv$ are pack and unpack bit fields. $(p \dashv m)$ means to **unpack** the string p using mask m. For example: $(1011 \dashv 110011)$ gives 100011. The length of p must be the same as the number of 1 bits in m. $(p \vdash m)$ means to **pack** the string p using mask m. For example: $(1011011 \vdash 1100111)$ gives 10011. The length of p must be the same as the length of m. The resulting string has the same number of bits as there are 1 bits in m. An alternate implementation of this operator may pad the resulting bitstring on the left with `0` so that the result is the same length as the two operands. The results of both forms contain equal information. All logic operators are bitwise operators. Overbar is the one's complement operator. The names of these theorems are my own.

Compressive Subset	$(p \vdash p)$	$=$	$(\vec{1} \vdash p)$
Inverse Property:	$(p \dashv m) \vdash m$	$=$	$p \wedge (\vec{1} \vdash m)$
SemiInverse Property:	$(p \vdash m) \dashv m$	$=$	$p \wedge m$
Associative Property:	$((p \dashv m) \dashv n)$	$=$	$(p \dashv (m \dashv n))$
Negation of Pack:	$\overline{p \vdash m}$	$=$	$(\overline{p} \vdash m) \vee \overline{(\vec{1} \vdash m)}$
Negation of Unpack:	$\overline{p \dashv m}$	$=$	$\overline{m} \vee (\overline{p} \dashv m)$
	$\overline{p \dashv m}$	$=$	$\overline{m} \oplus (\overline{p} \dashv m)$
Distributive Property:	$(p \vee q) \dashv m$	$=$	$(p \dashv m) \vee (q \dashv m)$
	$(p \oplus q) \dashv m$	$=$	$(p \dashv m) \oplus (q \dashv m)$
	$(p \wedge q) \dashv m$	$=$	$(p \dashv m) \wedge (q \dashv m)$
	$(p \vee q) \vdash m$	$=$	$(p \vdash m) \vee (q \vdash m)$
	$(p \oplus q) \vdash m$	$=$	$(p \vdash m) \oplus (q \vdash m)$
	$(p \wedge q) \vdash m$	$=$	$(p \vdash m) \wedge (q \vdash m)$
Destructive Distribution:	$(m \wedge p) \vdash p$	$=$	$m \vdash p$
	$(m \oplus p) \vdash p$	$=$	$\overline{m} \vdash p$
	$(m \dashv p) \wedge p$	$=$	$m \dashv p$
	$(m \dashv p) \oplus p$	$=$	$\overline{m} \dashv p$
	$(p \dashv (m \vee n)) \vdash m$	$=$	$p \vdash (m \vdash (m \vee n))$